Iteration Strategy for Upwind/Relaxation Solutions to the Thin-Layer Navier-Stokes Equations

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Abstract

An efficient iteration strategy for supersonic flow solutions to the parabolized and thin-layer Navier-Stokes equations is described. Modifications are made to a previously developed upwind/relaxation algorithm to suppress the upstream influence within the subsonic part of the boundary layer through restriction of the streamwise pressure gradient. A "parabolized" solution then is efficiently obtained by marching downstream and locally iterating each crossflow plane to convergence. The parabolized solution, fully conservative and second-order-accurate, provides an excellent solution for problems without strong adverse streamwise pressure gradients. If necessary, the thin-layer solution may be recovered by further iteration using the original upwind/relaxation algorithm.

Contents

Governing Equations/Spatial Discretization

The unsteady, compressible, thin-layer Navier-Stokes equations may be written in generalized coordinates as

$$\frac{\partial \hat{Q}}{\partial t}+\frac{\partial \hat{F}}{\partial \xi}+\frac{\partial \hat{G}}{\partial \eta}+\frac{\partial(\hat{H}-\hat{H}_v)}{\partial \zeta}=0 \tag{1}$$

where $Q=\hat{Q}J$ is the vector of conserved variables, J is the transformation Jacobian, and $\hat{F}$, $\hat{G}$, $\hat{H}$, and $\hat{H}_v$ are inviscid and viscous flux vectors in the streamwise (ξ), circumferential (η), and body-normal (ζ) coordinate directions. The equations are discretized and evaluated using a finite-volume approach, where metric terms are evaluated geometrically, to solve for a cell-centered, volume-averaged Q (i, j, k indices). The inviscid flux vectors $\hat{F}$, $\hat{G}$, $\hat{H}$, evaluated at cell interfaces ($i\pm 1/2$, $j\pm 1/2$, $k\pm 1/2$ indices), are split into positive and negative contributions according to the sign of the eigenvalues of the flux Jacobian matrices and differenced accordingly. For example,

$$(\delta_\xi \hat{F})_i=[\hat{F}^+(Q^-)+\hat{F}^-(Q^+)]_{i+\frac{1}{2}}$$
$$-[\hat{F}^+(Q^-)+\hat{F}^-(Q^+)]_{i-\frac{1}{2}} \tag{2}$$

in which the flux $\hat{F}$ is split according to the streamwise contravariant Mach number $M_\xi=U/(c|\nabla\xi|)$. For supersonic flow,

$$M_\xi \geq +1:\ \hat{F}^+=\hat{F}, \qquad \hat{F}^-=0 \tag{3a}$$

$$M_\xi \leq -1:\ \hat{F}^-=\hat{F}, \qquad \hat{F}^+=0 \tag{3b}$$

For subsonic flow, $|M_\xi|<1$, van Leer's flux vector splitting in three-dimensional generalized coordinates[1] is used. The Q^- and Q^+ are obtained from an upwind-biased interpolation of cell-centered values. No limiting was required in the present results, and fully upwind first- or second-order interpolation is given as, for example,

$$Q^-_{i+\frac{1}{2}}=Q_i+(\phi_\xi/2)(Q_i-Q_{i-1}) \tag{4a}$$

$$Q^+_{i+\frac{1}{2}}=Q_{i+1}-(\phi_\xi/2)(Q_{i+2}-Q_{i+1}) \tag{4b}$$

where

$$\phi_\xi=\begin{cases}0 \text{ (first order)}\\ 1 \text{ (second order)}\end{cases}$$

Differencing of the thin-layer viscous terms corresponds to second-order central differences.

Time Integration

Using a backward Euler time discretization, the governing equations are linearized and written in delta form. First-order implicit spatial differencing is employed, and second-order spatial differencing is used in the evaluation of the residual

$$R(Q^n)=-[\delta_\xi \hat{F}^n+\delta_\eta \hat{G}^n+\delta_\zeta(\hat{H}^n-\hat{H}^n_v)] \tag{5}$$

Relaxation is employed in the streamwise direction, and approximate factorization is employed in the crossflow plane. A steady-state solution is obtained by a series of alternating forward and backward streamwise relaxation sweeps as dictated by stability considerations. For a forward relaxation sweep, using a nonlinear residual update, the equation assumes the form

$$[N+\delta_\eta\partial\hat{G}/\partial Q]N^{-1}[N+\delta_\zeta(\partial\hat{H}/\partial Q-\partial\hat{H}_v/\partial Q)]\Delta Q^n$$
$$=R(Q^{n+1}_{i-2},Q^{n+1}_{i-1},Q^n_i,Q^n_{i+1},Q^n_{i+2}) \tag{6}$$

where

$$N=I/(J\Delta t)_i+(\partial\hat{F}^+/\partial Q)_{i+\frac{1}{2}}-(\partial\hat{F}^-/\partial Q)_{i-\frac{1}{2}}$$

Presented as Paper 87-1113 at the AIAA 8th Computational Fluid Dynamics Conference, Honolulu, HI, June 9-11, 1987; received July 13, 1987; revision received Sept. 5, 1988. Full paper available at the AIAA Library, 555 W. 57th St., New York, NY 10019.

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and, for clarity, only the streamwise spatial index is shown in the residual. A finite timestep for maximum damping exists with approximate factorization corresponding to a Courant number of $\mathcal{O}(10)$ in the inviscid field. A locally varying timestep is used to accelerate convergence.

Local Iteration

For streamwise supersonic flow, $M_\xi (Q^-)_{i\pm 1/2} > 1$, the flux evaluation involves only Q_{i-2}, Q_{i-1} and Q_i, since $\hat{F}^-_{i\pm 1/2} = 0$. If the interpolated streamwise interface Mach number is supersonic for all volumes within the ith crossflow plane, the solution is efficiently obtained by a single forward sweep (global iteration) with local iteration within each crossflow plane to reduce the corresponding crossflow plane residual to the desired convergence level.[2] For viscous flows, the boundary layer will contain a region of streamwise subsonic flow, and the above procedure is invalid. As shown by Vigneron et al.,[3] the negative eigenvalue associated with $\partial\hat{F}/\partial\xi$ can be eliminated by restricting the streamwise flux vector

$$\hat{F}_p = [\rho U, \rho U u + \omega p \nabla \xi, (e+p)U]^T \tag{7}$$

where

$$\omega = \min \{\sigma\gamma M_\xi^2/[1 + (\gamma - 1)\, M_\xi^2]\ 1\}$$

and ρ, p, and e are the density, pressure, and total energy, respectively, $\boldsymbol{u}$ is the Cartesian velocity vector, U is the streamwise contravariant velocity, and σ is a safety factor ($\sigma < 1$). A parabolized solution then is obtained by splitting the streamwise flux (and flux Jacobian) as

$$\hat{F}^+ (Q^-)_{i\pm 1/2} = \hat{F}_p (Q^-)_{i\pm 1/2}, \qquad \hat{F}^- (Q^+)_{i\pm 1/2} = 0 \tag{8}$$

and employing local iteration. For a nonlinear residual update, the residual then has the form

$$R = R(Q_{i-2}^{n+1}, Q_{i-1}^{n+1}, Q_i^\ell)$$

Since Q_{i-2}^{n+1} and Q_{i-1}^{n+1} are converged upstream solutions, local iteration in the crossflow plane is required to update Q_i as

$$Q_i^{\ell+1} = Q_i^\ell + \Delta Q_i^\ell$$

where superscript ℓ denotes the ℓth local iteration of the ith crossflow plane and, at convergence, $\Delta Q_i^\ell \to 0$, $Q_i^{\ell+1} \to Q_i^{n+1}$. The Vigneron approach provides an excellent approximation to the thin-layer Navier-Stokes equations for favorable streamwise pressure gradients. However, it is well-known that such an approach may break down for adverse pressure gradients, "marching" over what should be a region of streamwise separation. The local iteration strategy then provides a good initial condition for subsequent iterations with the global scheme to efficiently recover the separated flow solution.

Results

Only results for the laminar flow about an elliptical body missile at a flight condition of Mach 2.5, 20-deg angle of attack, and a Reynolds number based on body length of 4.66 million are presented. The missile was a sharp-tipped, Adams minimum drag body with a 3:1 elliptical cross section. Numerical solutions were obtained on a Cyber 205 computer using a vectorized code on a grid of $33 \times 51 \times 61$ points. The convergence criterion was set as a four-order-of-magnitude reduction in the global and crossflow residual L_2 norms for the global and local iteration schemes, respectively. Virtually identical results were obtained with the two methods, as would be expected for a favorable streamwise pressure gradient. Figure 1 shows a comparison of the two solutions for the predicted density contours at the downstream end of the missile body. The flow at this station is characterized by a strong primary vortex and two smaller secondary vortices on each side of the body. Table 1 shows the modified algorithm to be roughly 18 times faster than the more general global algorithm, where one iteration is defined as a single forward or backward sweep through the mesh. This reduction is due to the inherently better efficiency of the algorithm for this class of flow (factor of 9) and the use of "frozen" flux Jacobian and lower-upper block tridiagonal decompositions in the crossflow plane local iterations of the modified scheme (factor of 2). This later procedure was not feasible in the original scheme because of the much larger memory requirements.

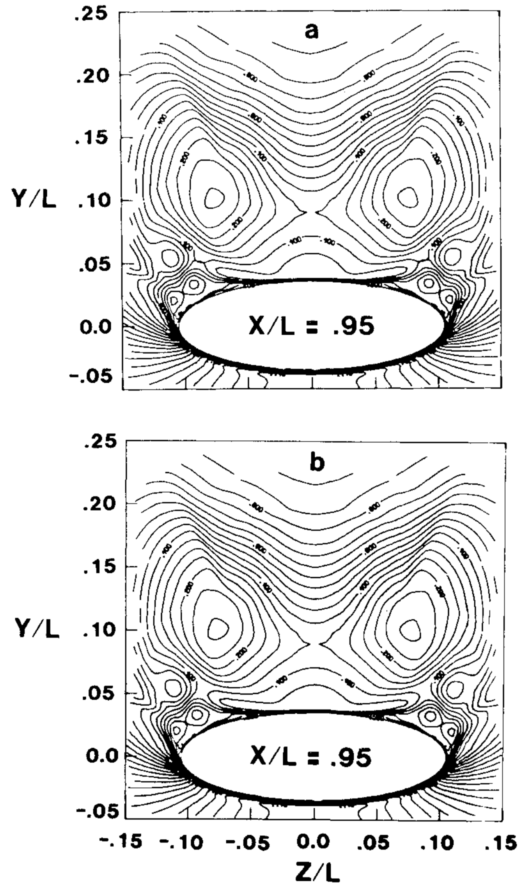

Fig. 1 Comparison of density contours for a) local (parabolized) and b) global (thin-layer) iteration solutions, elliptic body missile, $M = 2.5$, $\alpha = 20$ deg.

Table 1 Relative efficiency of iteration strategies for elliptical body missile

Strategy	Iterations	cpu time, s
Local	1	206
Global	440	3707

References

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[3]Vigneron, Y. C., Rakich, J. V., and Tannehill, J. C., "Calculation of Supersonic Viscous Flow Over Delta Wings with Sharp Leading Edges," AIAA Paper 78-1137, July 1978.